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On the solution of fractional evolution equations

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Abstract

This paper is devoted to the solution of the bi-fractional differential equation

$$\left({}^{C}D_t^{\alpha}u\right)(t,x) = \lambda\left({}^{L}D_x^{\beta}u\right)(t,x) \qquad (t>0, -\infty < x < \infty)$$

for real $0 < \alpha \leqslant 1$, $\beta > 0$ and $\lambda \neq 0$, with the initial conditions

$$\lim_{x\to\pm\infty} u(t,x) = 0 \qquad u(0+,x) = g(x).$$

Here $\left({}^{C}D_t^{\alpha}u\right)(t,x)$ is the partial derivative coinciding with the Caputo fractional derivative for $0 < \alpha < 1$ and with the usual derivative for $\alpha = 1$, while $\left({}^{L}D_x^{\beta}u\right)(t,x))$ is the Liouville partial fractional derivative $\left({}^{L}D_t^{\beta}u\right)(t,x))$ of order $\beta > 0$. The Laplace and Fourier transforms are applied to solve the above problem in closed form. The fundamental solution of these problems is established and its moments are calculated. The special case $\alpha = 1/2$ and $\beta = 1$ is presented, and its application is given to obtain the Dirac-type decomposition for the ordinary diffusion equation.

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1. Introduction

This paper deals with the solution of the linear fractional differential equation

$$\left({}^{C}D_t^{\alpha}u\right)(t,x) = \lambda\left({}^{L}D_x^{\beta}u\right)(t,x) \qquad (t>0, -\infty<x<\infty, 0<\alpha<1, \beta>0) \tag{1}$$

with $0 < \alpha \leqslant 1$, $\beta > 0$ and real $\lambda \in \mathbb{R} = (-\infty, \infty)$, $\lambda \neq 0$. Here $\left({}^{C}D_t^{\alpha}u\right)(t,x)$ is the partial derivative defined by

$$\left({}^{C}D_t^{\alpha}u\right)(t,x) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{u(\tau,x)-u(0,x)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau \tag{2}$$

for $0 < \alpha < 1$, $\Gamma(z)$ being the Gamma function [3, section 24.2], and by

$$\left({}^{C}D_x^1 u\right)(t, x) = \frac{\partial u(t, x)}{\partial t} \tag{3}$$

for $\alpha = 1$, while $\left({}^{L}D_x^{\beta} u\right)(t, x)$ is the so-called Liouville partial fractional derivative of order $\beta > 0$ defined by

$$\left({}^{L}D_x^{\beta} u\right)(t, x) = \frac{1}{\Gamma(m-\beta)}\left(\frac{\partial}{\partial x}\right)^m \int_{-\infty}^{x} \frac{u(t, y)}{(x-y)^{\beta-m+1}}\partial y \qquad (x \in \mathbb{R}, m = -[-\beta]) \tag{4}$$

$[\beta]$ being the integer part of β; see [14, section 24.2].

A one-dimensional fractional derivative of the form (2) defined by

$$\left({}^{C}D_t^{\alpha} f\right)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{f(\tau) - f(0)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau \qquad (0 < \alpha < 1) \tag{5}$$

is known as the Caputo fractional derivative of order α (for example, see [13], equation 2.138). Therefore we call the derivative in (2) the partial Caputo fractional derivative. Such a Caputo fractional differential operator can be considered as a regularized version of the Riemann–Liouville fractional differential operator. In particular, if $f(x)$ is continuously differentiable, (5) takes the form

$$\left({}^{C}\mathcal{D}_t^{\alpha} f\right)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau \qquad (0 < \alpha < 1) \tag{6}$$

which is often used as the definition of the Caputo fractional derivative.

When $\beta = 1$,

$$\left({}^{L}D_x^1 u\right)(t, x) = \frac{\partial u(t, x)}{\partial x} \tag{7}$$

and equation (1) takes the form

$$\left({}^{C}D_t^{\alpha} u\right)(t, x) = \lambda\frac{\partial u(t, x)}{\partial x} \qquad (t > 0, x \in \mathbb{R}; 0 < \alpha \leqslant 1). \tag{8}$$

Such an equation with $\alpha = 1/2$ arises in the analysis of diffusion mechanisms with internal degrees of freedom while studying the square root of the one-dimensional diffusion equation $u_t - u_{xx} = 0$; see [18, 19].

We note that the bi-fractional differential equations of type (1), with particular values of the parameters α and β and different fractional derivatives, appear in a natural way to model the dynamics of processes involving different scales of space and/or time and in the theory of complex systems in many branches of applied sciences and engineering. In particular, the fractional differential operators have been used as suitable tools for mathematical modelling of anomalous diffusion (sub- and super-diffusion) through the use of the well-known continuous time random walk (CTRW) method, the Lévy stable distributions, the generalized central limit and the Laplace and Fourier integral transforms (see, for instance, [17, 9, 7, 10]).

We must stress that there are many different fractional differential operators which generalize the ordinary one. From the point of view of applications, the main property of these fractional operators is that they are non-local.

We must point out here that there are several pathways to the use of fractional models of the form (1) in different applied fields; for instance, see [9, 7]. In particular, in the case when mathematical models are connected with anomalous diffusion we could use the CTRW approach introduced by Montroll [11], or through the Langevin equation approach (see [5]), or by generalization of the classical first Fick's law combined with the conservation law (for example, see [12]) and others. When we introduce in the ordinary diffusion equation a time-fractional derivative, such as the Riemann–Liouville or the Caputo, we obtain good models for the sub-diffusion processes, but not for the case of super-diffusion. The inversion of the well-known Riesz fractional integration operator (for example, see [14, section 25]) is probably the best operator that can be used to generalize the space derivative to the model of super-diffusion problems keeping the symmetrical property of the fundamental solution of bi-fractional equations in connection with the CTRW method.

Here we use the Liouville operator as the space-fractional derivative because we would like to apply the solution of equation (1) to the Dirac-type decomposition of the ordinary diffusion equation. The Riesz fractional derivative of order $\alpha > 0$ is not suitable for such a purpose, because such a derivative, represented by a hypersingular integral, in the case $\alpha = 1$, generally speaking, does not coincide with the usual derivative; for example, see [14, sections 25–26].

In this paper we study the boundary value problem for equation (1) with the conditions

$$\lim_{x\to\pm\infty} u(t,x) = 0 \qquad u(0+,x) = g(x). \tag{9}$$

This paper is organized as follows. Section 2 is devoted to obtaining the solution in closed form of the problem (8), (9). Using the Laplace and the Fourier transforms we deduce the explicit solution of this problem in terms of Mittag–Leffler function $E_\alpha(z)$; for example, see [4, section 18.1]. In section 3 we show that the fundamental solution of the problem (8), (9) can be expressed in terms of the Wright function $\varphi(\alpha,\beta;z)$ defined for complex $z \in \mathbb{C}$, $\alpha > -1$ and $\beta \in \mathbb{R}$ by the series

$$\varphi(\alpha,\beta;z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)n!} \tag{10}$$

(see [4, 18.1(27)]), and we evaluate the moments of this fundamental solution. Section 4 deals with the explicit solution of the problem (1), (9) and with the evaluation of its moments. In section 5 we present the special case of the problem (8), (9) with $\alpha = 1/2$ and $\beta = 1$. Finally, we apply this result to obtain the explicit solution of a Dirac-type decomposition of the ordinary diffusion equation.

2. Solution of the time-fractional problem

To obtain the explicit solution of the problem (8), (9) we shall use the well-known Laplace transform of a function $u(t,x)$ with respect to t:

$$(\mathcal{L}_t u)(s,x) = \int_0^\infty \mathrm{e}^{-st} u(t,x)\,\mathrm{d}t \tag{11}$$

for any fixed $x \in \mathbb{R}$, and the Fourier transform with respect to x:

$$(\mathcal{F}_x u)(t,k) = \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}kx} u(t,x)\,\mathrm{d}x \tag{12}$$

for any fixed $t \in \mathbb{R}_+$, and the inverse Laplace transform with respect to s:

$$\left(\mathcal{L}_s^{-1} u\right)(t,x) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma+\mathrm{i}\infty}^{\gamma-\mathrm{i}\infty} \mathrm{e}^{st} u(s,x)\,\mathrm{d}s \tag{13}$$

with a fixed $\gamma \in \mathbb{R}$, and the inverse Fourier transform with respect to k:

$$\left(\mathcal{F}_k^{-1} u\right)(t,x) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}kx} u(t,k)\,\mathrm{d}k \tag{14}$$

together with the relations

$$\left(\mathcal{L}_s^{-1}\mathcal{L}_t u\right)(t,x) = u(t,x) \qquad \left(\mathcal{F}_k^{-1}\mathcal{F}_x u\right)(t,x) = u(t,x). \tag{15}$$

We shall use (11)–(15) in spaces of classical and generalized functions. The characterization of classical functions, for which the above one-dimensional direct and inverse Laplace and Fourier transforms exist and the relations in (15) hold, can be found, for example, in the books by Dithin and Prudnikov [2, chapters 1, 2] and Sneddon [16, sections 3–4]. We only indicate that the integrals in (11)–(14) are understood, as usual, in the sense of principal value, and the real constant γ in (13) can be chosen such that $\gamma > \sigma_c$, where σ_c is the so-called abscissa of convergence of the integral (11); see [2, chapter 2, section 1].

We denote by $\mathcal{LF} = \mathcal{L}(\mathbb{R}_+) \times \mathcal{F}(\mathbb{R})$, $\mathbb{R}_+ = (0, \infty)$, the space of functions $u(t, x)$ such that there exist the Laplace transform (11) and the Fourier transform (12), and we shall use the following notation:

$$\hat{u}(s, k) \equiv (\mathcal{F}_x \mathcal{L}_t u)(s, k) = \int_{-\infty}^{\infty} \int_0^{\infty} \mathrm{e}^{-st} \, \mathrm{e}^{\mathrm{i}kx} u(t, x) \, \mathrm{d}t \, \mathrm{d}x \qquad (t > 0). \quad (16)$$

Lemma 1. *Let $g(x)$ be a function such that there exists the Fourier transform $G(k)$:*

$$G(k) = (\mathcal{F}g)(k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kt} g(t) \, \mathrm{d}t \qquad (k \in \mathbb{R}). \quad (17)$$

Then the solution $u(x, t) \in \mathcal{LF}$ of the problem (8), (9) is given by the formula

$$u(t, x) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \mathrm{e}^{st} s^{\alpha - 1} \, \mathrm{d}s \, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(k)}{s^\alpha + \mathrm{i}\lambda k} \, \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}k \quad (18)$$

provided that the integral on the right-hand side exists.

Proof. We apply the Laplace transform (11) to equation (8) and use the following formula for the Laplace transform of the Caputo derivative (2) [13, (2.140)]:

$$\left(\mathcal{L}_t \, {}^C D_t^\alpha u\right)(s, x) = s^\alpha (\mathcal{L}_t u)(s, x) - s^{\alpha - 1} u(0+, x) \qquad (0 < \alpha \leqslant 1) \quad (19)$$

where s^α is understood, as usually, as the corresponding value of the main branch of the analytic function s^α in the complex plane s with the cut along the positive half-axis $\mathbb{R}_+$. Note that (19) yields the known formula for the Laplace transform of the usual derivative when $\alpha = 1$. Making use of such an application of the Laplace transform (11) to (8) and taking the condition $u(0+, x) = g(x)$ into account, we have

$$s^\alpha (\mathcal{L}_t u)(s, x) - s^{\alpha - 1} g(x) = \lambda \frac{\partial}{\partial x} (\mathcal{L}_t u)(s, x). \quad (20)$$

Applying the Fourier transform (12) and using the formula for the Fourier transform of the ordinary derivatives

$$(\mathcal{F}_x D_x v)(s, k) = (-\mathrm{i}k)(\mathcal{F}_x v)(s, k) \qquad \left(D_x = \frac{\partial}{\partial x}\right) \quad (21)$$

in accordance with (12) and (16) we obtain

$$s^\alpha \hat{u}(s, k) - s^{\alpha - 1} G(k) = (-\mathrm{i}\lambda k) \hat{u}(s, k). \quad (22)$$

From here we deduce the following relation:

$$\hat{u}(s, k) = \frac{s^{\alpha - 1} G(k)}{s^\alpha + \mathrm{i}\lambda k}. \quad (23)$$

Applying to (23) the inverse Laplace and inverse Fourier transforms and taking (13)–(15) into account, we obtain the solution (18) of the problem (8), (9). □

Remark 1. If $G(k)$ satisfy some additional conditions, the inner integral in (18) can be evaluated by using the residue theory; for example, see [2, chapter 2, section 1].

The next assertion gives another representation for the solution $u(t, x)$ of the problem (8), (9) in terms of the special Mittag–Leffler function $E_\alpha(z)$ defined for $z, \alpha \in \mathbb{C}$ by

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)} \tag{24}$$

see [4, section 18.1].

Lemma 2. *Let $g(x)$ be a function such that there exists its Fourier transform $G(k)$. Then the solution $u(x, t) \in \mathcal{LF}$ of the problem (8), (9) has the form*

$$u(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_\alpha(-\mathrm{i}\lambda k t^\alpha) G(k)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}k \tag{25}$$

provided that the integral on the right-hand side exists.

Proof. It is known that the Laplace transform of the Mittag–Leffler function $E_\alpha(\mu t^\alpha)$ is given by the formula

$$(\mathcal{L} E_\alpha(\mu t^\alpha))(s) = \frac{s^{\mu-1}}{s^\alpha - \mu} \qquad (|\mu s^{-\alpha}| < 1) \tag{26}$$

for example, see [4, section 18.1]. Applying to (23) the inverse Laplace transform and using the first relation in (15) and (26) with $\mu = -\mathrm{i}\lambda k$, we have

$$(\mathcal{F}_x u)(t, k) = E_\alpha(-\mathrm{i}\lambda k t^\alpha) G(k). \tag{27}$$

Then, the application of the inverse Fourier transform to (27) and the second formula in (15) yields the explicit solution (25).

The final assertion in this section presents a solution of the problem (8), (9) for the analytic function $g(x)$. □

Lemma 3. *Let $g(x)$ be an analytic function of the real variable x such that*

$$\lim_{|x|\to\infty} g^{(j)}(x) = 0 \qquad (j = 0, 1, 2, \ldots). \tag{28}$$

Then the solution $u(t, x)$ of the problem (8), (9) is given by

$$u(t, x) = \sum_{j=0}^{\infty} \frac{(\lambda t^\alpha)^j}{\Gamma(\alpha j + 1)} g^{(j)}(x) \tag{29}$$

provided that the series in (29) converges for any $x \in \mathbb{R}$ and any $t > 0$.

Proof. Substituting (24) into (25) and interchanging the order of integration and series (which is possible by the uniform convergence of series represented by the entire Mittag–Leffler function), we have

$$\begin{aligned} u(t, x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\sum_{j=0}^{\infty} \frac{(-\mathrm{i}\lambda k t^\alpha)^j}{\Gamma(\alpha j + 1)} \right] G(k)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}k \\ &= \sum_{j=0}^{\infty} \frac{(\lambda t^\alpha)^j}{\Gamma(\alpha j + 1)} \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-\mathrm{i}k)^j G(k)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}k. \end{aligned} \tag{30}$$

Using the known formula

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} (-\mathrm{i}k)^j G(k)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}k = \left(\mathcal{F}_k^{-1} G\right)^{(j)}(x) = g^{(j)}(x) \qquad (j = 1, 2, \ldots) \tag{31}$$

from (30) and (31) we deduce the representation for the solution $u(t,x)$ of the problem (8), (9) in the form (29). □

Remark 2. Another proof of lemma 3 can be given by the substitution of (29) into (8) and carrying out term by term fractional differentiation, which is possible by the analyticity of $g(x)$ and the uniform convergence of the power series of t represented by the Mittag–Leffler function (24).

Example 1. The conditions of lemma 3 hold for the following problem:

$$\begin{aligned} &\left({}^{C}D_t^{\alpha}u\right)(t,x)=\lambda\frac{\partial u(t,x)}{\partial x} \qquad (t>0, x\in\mathbb{R}) \\ &\lim_{|x|\to\infty} u(t,x)=0 \qquad u(0+,x)=\mathrm{e}^{-\mu|x|} \quad (\mu>0) \end{aligned} \tag{32}$$

and its solution is given by

$$u(t,x)=\mathrm{e}^{-\mu|x|}E_{\alpha}(-\mu\lambda t^{\alpha}). \tag{33}$$

Example 2. If the conditions of lemma 3 are satisfied, then the problem

$$\frac{\partial u(t,x)}{\partial t}=\lambda\frac{\partial u(t,x)}{\partial x} \qquad (t>0, x\in\mathbb{R}) \qquad \lim_{x\to\pm\infty} u(t,x)=0 \quad u(0+,x)=g(x) \tag{34}$$

with $\lambda\in\mathbb{R}$ has the well-known explicit solution

$$u(t,x)=g(x+\lambda t). \tag{35}$$

3. Fundamental solution of the time-fractional problem

First of all we note that the method for the solution of the initial value problem (8), (9) in the space $\mathcal{LF}=\mathcal{L}(\mathbb{R}_+)\times\mathcal{F}(\mathbb{R})$, used in section 2 and based on the Laplace and Fourier transforms, can also be applied in the space $\mathcal{LF}'=\mathcal{L}(\mathbb{R}_+)\times\mathcal{F}'(\mathbb{R})$, where $\mathcal{F}'(\mathbb{R})$ is a space of Fourier transform of the generalized function, if we replace the Fourier transform in (12) by the corresponding Fourier transform of generalized functions. For example, we can use any of the well-known spaces $\mathcal{F}'(\mathbb{R})=S'$ or $\mathcal{F}'(\mathbb{R})=D'$. The Fourier transform in the spaces S' and D' was introduced by Schwartz [15] and Gelfand and Shilov [6], respectively. In this connection see also the books by Brychkov and Prudnikov [1], Vladimirov [20] and Zemanian [21].

Thus we can consider the initial value problem of the form (8), (9), in which $g(x)$ is replaced by the Dirac delta function $\delta(x)$

$$\begin{aligned} &\left({}^{C}D_t^{\alpha}u\right)(t,x)=\lambda\frac{\partial u(t,x)}{\partial x} \qquad (t>0, x\in\mathbb{R}) \qquad 0<\alpha\leqslant 1 \\ &\lim_{|x|\to\infty} u(t,x)=0 \qquad u(0+,x)=\delta(x). \end{aligned} \tag{36}$$

The solution $u(t,x)$ of this problem is known as the fundamental solution.

Theorem 1. *The fundamental solution $u(t,x)\in\mathcal{LF}'$ of the problem (36) is given by*

$$u(t,x)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{e}^{st}s^{\alpha-1}\,\mathrm{d}s\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{s^{\alpha}+\mathrm{i}\lambda k}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \qquad (\gamma\in\mathbb{R}). \tag{37}$$

Moreover the following function also represents the fundamental solution of the problem (36):

$$u(t,x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}E_{\alpha}(-\mathrm{i}\lambda k t^{\alpha})\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \tag{38}$$

provided that the integrals on the right-hand sides of (37) and (38) exist.

Proof. It is known that

$$\delta(x) \in D' \qquad \text{and} \qquad (\mathcal{F}_x\delta)(k) = 1 \tag{39}$$

see, for example, [1, chapter 2, section 2 and chapter 8, section 7, no 662]. Then $G(k) = 1$ and using the same arguments as in the proofs of lemmas 1 and 2 we obtain the solutions $u(t, x)$ in the forms (18) and (25) with $G(k) = 1$, which yield the fundamental solution of the problem (36) in the forms (37) and (38), respectively. □

Corollary 1. *The fundamental solution of the problem (36) has the explicit form*

$$u(t,x) = \begin{cases} 0 & x > 0 \\ -\dfrac{\mathrm{i}}{2\pi\lambda}\displaystyle\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st} s^{\alpha-1}\,\mathrm{e}^{s^\alpha x/\lambda}\,\mathrm{d}s & x < 0 \end{cases} \tag{40}$$

when $\lambda > 0$*, while for* $\lambda < 0$

$$u(t,x) = \begin{cases} \dfrac{\mathrm{i}}{2\pi\lambda}\displaystyle\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st} s^{\alpha-1}\,\mathrm{e}^{s^\alpha x/\lambda}\,\mathrm{d}s & x > 0 \\ 0 & x < 0. \end{cases} \tag{41}$$

Moreover, these expressions mean that the problem (36) has the fundamental solution $u(t, x) = 0$ *in the cases* $x > 0$, $\lambda > 0$ *and* $x < 0$, $\lambda < 0$.

Proof. We apply the residue theory to evaluate the inner integral in (37) (see, for example, [2, chapter 2, section 1]). If $x < 0$, then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{s^\alpha + \mathrm{i}\lambda k}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k &= 2\pi\mathrm{i}\operatorname*{res}_{k=\mathrm{i}s^\alpha/\lambda}\left[\frac{1}{s^\alpha + \mathrm{i}\lambda k}\,\mathrm{e}^{-\mathrm{i}kx}\right] \\ &= \begin{cases} \dfrac{2\pi}{\lambda}\,\mathrm{e}^{s^\alpha x/\lambda} & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}\end{aligned} \tag{42}$$

while for $x > 0$

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{s^\alpha + \mathrm{i}\lambda k}\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k &= -2\pi\mathrm{i}\operatorname*{res}_{k=\mathrm{i}s^\alpha/\lambda}\left[\frac{1}{s^\alpha + \mathrm{i}\lambda k}\,\mathrm{e}^{-\mathrm{i}kx}\right] \\ &= \begin{cases} 0 & \lambda > 0 \\ -\dfrac{2\pi}{\lambda}\,\mathrm{e}^{s^\alpha x/\lambda} & \lambda < 0. \end{cases}\end{aligned} \tag{43}$$

Substituting this relation into (37) we find the solution of the problem (36) in the forms (40) and (41). □

Corollary 2. *The fundamental solution of the problem (36) is given in terms of the Wright function (10) by*

$$u(t,x) = \begin{cases} 0 & x > 0 \\ \dfrac{1}{\lambda t^\alpha}\varphi\left(-\alpha, 1-\alpha; \dfrac{x}{\lambda t^\alpha}\right) & x < 0 \end{cases} \tag{44}$$

for $\lambda > 0$ *and*

$$u(t,x) = \begin{cases} -\dfrac{1}{\lambda t^\alpha}\varphi\left(-\alpha, 1-\alpha; \dfrac{x}{\lambda t^\alpha}\right) & x > 0 \\ 0 & x < 0 \end{cases} \tag{45}$$

for $\lambda < 0$.

Proof. We use the notation

$$v(t,x) = \frac{\mathrm{i}}{2\pi\lambda}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st} s^{\alpha-1}\,\mathrm{e}^{s^\alpha x/\lambda}\,\mathrm{d}s. \tag{46}$$

Expanding the exponential function $\mathrm{e}^{s^\alpha x/\lambda}$ in a Taylor series and interchanging the order of integration and series (which is possible by the uniform convergence of the exponential series), we have

$$v(t,x) = \frac{\mathrm{i}}{2\pi\lambda}\sum_{n=0}^{\infty}\left(\frac{x}{\lambda}\right)^n \frac{1}{n!}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} s^{\alpha n+\alpha-1}\,\mathrm{e}^{st}\,\mathrm{d}s \qquad (\gamma\in\mathbb{R}). \tag{47}$$

Performing the substitution $st = \sigma$, transforming the contour into the Hankel contour [3, section 1.6] and using the Hankel representation for the Gamma function [3, section 1.6 (2)]

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{0+} \mathrm{e}^{t} t^{-z}\,\mathrm{d}t \qquad |\arg(t)| \leqslant \pi \tag{48}$$

we obtain

$$v(t,x) = -\frac{1}{\lambda t^\alpha}\sum_{n=0}^{\infty}\frac{1}{\Gamma(-\alpha n+1-\alpha)n!}\left(\frac{x}{\lambda t^\alpha}\right)^n. \tag{49}$$

Taking (10) into account, we deduce the explicit representation for the $v(t,x)$ in terms of the Wright function:

$$v(t,x) = -\frac{1}{\lambda t^\alpha}\varphi\left(-\alpha, 1-\alpha; \frac{x}{\lambda t^\alpha}\right). \tag{50}$$

Thus the fundamental solution in (40) and (41) takes the forms (44) and (45), respectively. □

Sometimes the Wright function (10) is represented by

$$W(z;\alpha,\beta) = \varphi(\alpha,\beta;z) \tag{51}$$

for example, see [13, (1.156)]. Then (50) can be rewritten as

$$v(t,x) = -\frac{1}{\lambda t^\alpha} W\left(\frac{x}{\lambda t^\alpha}; -\alpha, 1-\alpha\right) \tag{52}$$

and corollary 2 can be reformulated as follows:

Corollary 3. *The fundamental solution of the problem (36) is given by*

$$u(t,x) = \begin{cases} 0 & x > 0 \\ \dfrac{1}{\lambda t^\alpha} W\left(\dfrac{x}{\lambda t^\alpha}; -\alpha, 1-\alpha\right) & x < 0 \end{cases} \tag{53}$$

and

$$u(t,x) = \begin{cases} -\dfrac{1}{\lambda t^\alpha} W\left(\dfrac{x}{\lambda t^\alpha}; -\alpha, 1-\alpha\right) & x > 0 \\ 0 & x < 0 \end{cases} \tag{54}$$

for $\lambda > 0$ *and* $\lambda < 0$*, respectively.*

We can also use the special case of (51) in the form [13, (1.160)]

$$M(z;\alpha) = W(-z;-\alpha,1-\alpha) = \sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(-\alpha n+1-\alpha)n!} \tag{55}$$

which leads to

Corollary 4. *The fundamental solution of the problem (36) is represented by*

$$u(t,x) = \begin{cases} 0 & x > 0 \\ \dfrac{1}{\lambda t^\alpha} M\left(-\dfrac{x}{\lambda t^\alpha};\alpha\right) & x < 0 \end{cases} \tag{56}$$

and

$$u(t,x) = \begin{cases} -\dfrac{1}{\lambda t^\alpha} M\left(-\dfrac{x}{\lambda t^\alpha};\alpha\right) & x > 0 \\ 0 & x < 0. \end{cases} \tag{57}$$

for $\lambda > 0$ *and* $\lambda < 0$*, respectively.*

Now we calculate the moments of the fundamental solution, by using the well-known property:

$$\int_{-\infty}^{+\infty} x^n u(t,x)\,\mathrm{d}x = (-\mathrm{i})^n \left[\frac{\mathrm{d}^n}{\mathrm{d}k^n}(\mathcal{F}_x u)(t,k)\right]_{k=0} \qquad (n = 0, 1, 2, \ldots) \tag{58}$$

and relation (27) with $G(k) = 1$

$$(\mathcal{F}_x u)(t,k) = E_\alpha(-\mathrm{i}\lambda k t^\alpha). \tag{59}$$

Substituting (59) into (58) and taking (24) into account we calculate the moments as follows:

$$\begin{aligned} \int_{-\infty}^{+\infty} x^n u(t,x)\,\mathrm{d}x &= (-\mathrm{i})^n \left[\frac{\mathrm{d}^n}{\mathrm{d}k^n}\sum_{j=0}^{\infty}\frac{(-\mathrm{i}\lambda k)^j t^{\alpha j}}{\Gamma(\alpha j+1)}\right]_{k=0} \\ &= (-\mathrm{i})^n \left[\sum_{j=n}^{\infty}\frac{(-\mathrm{i}\lambda)^j k^{j-n} t^{\alpha j}}{\Gamma(\alpha j+1)}\frac{\Gamma(j+1)}{\Gamma(j-n+1)}\right]_{k=0} \end{aligned}$$

and hence

$$\int_{-\infty}^{+\infty} x^n u(t,x)\,\mathrm{d}x = (-\lambda t^\alpha)^n \frac{\Gamma(n+1)}{\Gamma(\alpha n+1)} \qquad (n = 0, 1, 2, \ldots). \tag{60}$$

From here we deduce the formula for the moments of even order:

$$\int_{-\infty}^{+\infty} x^{2n} u(t,x)\,\mathrm{d}x = \frac{\Gamma(2n+1)}{\Gamma(2\alpha n+1)}(\lambda t^\alpha)^{2n} \qquad (n = 0, 1, 2, \ldots). \tag{61}$$

Remark 3. Formula (61) is also true for the even moments of the fundamental solution

$$u(t,x) = \frac{1}{2\lambda t^\alpha} M\left(\frac{|x|}{\lambda t^\alpha};\alpha\right) \tag{62}$$

of the initial problems for the fractional diffusion-wave equation

$$\begin{aligned} &\left({}^c D_t^{2\alpha} u\right)(t,x) = \lambda^2 \frac{\partial^2}{\partial x^2} u(t,x) \qquad (t > 0, x \in \mathbb{R}, 0 < \alpha \leqslant 1) \\ &\lim_{|x|\to\infty} u(t,x) = 0 \qquad u(0+,x) = \delta(x) \qquad u_t(0+,x) = 0. \end{aligned} \tag{63}$$

Such a result was obtained by Mainardi [8].

4. Solution of general problem

Now we study the general problem (1), (9) seeking the solution in the space $\mathcal{LF} = \mathcal{L}(\mathbb{R}_+) \times \mathcal{F}(\mathbb{R}), \mathbb{R}_+ = (0, \infty)$. The following result generalizing the ones in lemmas 1 and 2 holds.

Lemma 4. *Let $g(x)$ be a function such that there exists the Fourier transform $G(k)$ in (17). Then the solution $u(x, t) \in \mathcal{LF}$ of the problem (1), (9) has the form*

$$u(t, x) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st} s^{\alpha-1}\,\mathrm{d}s \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(k)}{s^\alpha - \lambda(-\mathrm{i}k)^\beta}\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \qquad (\gamma \in \mathbb{R}) \tag{64}$$

and also

$$u(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(k) E_\alpha(\lambda(-\mathrm{i}k)^\beta t^\alpha)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \tag{65}$$

provided that the integrals on the right-hand sides of (64) and (65) exist.

Proof. We shall use the formula for the Fourier transform of the Liouville derivative of order $\beta > 0$ [14, (7.4)]:

$$\left(\mathcal{F}_x\left({}^L D_x^\beta u\right)\right)(t, k) = (-\mathrm{i}k)^\beta (\mathcal{F}u)(t, k) \qquad (\beta > 0). \tag{66}$$

Applying (11) and (12) to equation (1) and taking into account the condition $u(0+, x) = g(x)$ and (66), we obtain the relation of the form (23) for $\hat{u}(s, k)$:

$$\hat{u}(s, k) = \frac{s^{\alpha-1} G(k)}{s^\alpha - \lambda(-\mathrm{i}k)^\beta}. \tag{67}$$

Using the inverse Laplace and Fourier transforms and taking the same arguments, as was done in the proofs of lemmas 1 and 2 in section 2, we obtain the solution $u(t, x)$ of the problem (1), (9) in the forms (64) and (65), respectively. □

Theorem 2. *The fundamental solution $u(t, x) \in \mathcal{LF}'$ of the problem*

$$\begin{aligned} &\left({}^C D_t^\alpha u\right)(t, x) = \lambda\left({}^L D_x^\beta u\right)(t, x) \qquad (t > 0, -\infty < x < \infty, 0 < \alpha < 1, \beta > 0) \\ &\qquad \lim_{|x|\to\infty} u(t, x) = 0 \qquad u(0+, x) = \delta(x) \end{aligned} \tag{68}$$

is given by

$$u(t, x) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st} s^{\alpha-1}\,\mathrm{d}s \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{s^\alpha - \lambda(-\mathrm{i}k)^\beta}\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \qquad (\gamma \in \mathbb{R}) \tag{69}$$

and also by

$$u(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_\alpha(\lambda(-\mathrm{i}k)^\beta t^\alpha)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \tag{70}$$

provided that the integrals on the right-hand sides of (69) and (70) exist.

Proof. By (39), $G(k) = 1$, and the results in (69) and (70) follow from (64) and (65), respectively.

Now, using formula (58), we calculate the moments of the fundamental solution. By (70), there holds the relation of the form (59):

$$(\mathcal{F}_x u)(t, k) = E_\alpha(\lambda(-\mathrm{i}k)^\beta t^\alpha). \tag{71}$$

Substituting this result into (58), we have for each $n \in \mathbb{N}$

$$\int_{-\infty}^{+\infty} x^n u(t,x)\,\mathrm{d}x = (-\mathrm{i})^n \left[\frac{\mathrm{d}^n}{\mathrm{d}k^n} \sum_{j=0}^{\infty} \frac{(-\mathrm{i}k)^{\beta j}(\lambda t^{\alpha})^j}{\Gamma(\alpha j+1)} \right]_{k=0}$$

$$= (-\mathrm{i})^n \left[\sum_{j=1}^{\infty} \frac{(-\mathrm{i})^{\beta j} k^{\beta j-n}(\lambda t^{\alpha})^j \Gamma(\beta j+1)}{\Gamma(\alpha j+1)\Gamma(\beta j-n+1)} \right]_{k=0}. \tag{72}$$

Then we obtain

$$\text{if} \quad \beta \neq r \qquad \int_{-\infty}^{+\infty} x^n u(t,x)\,\mathrm{d}x = \begin{cases} 0 & n < \beta \\ \nexists & n > \beta \end{cases} \tag{73}$$

$$\text{if} \quad \beta = r \qquad \int_{-\infty}^{+\infty} x^n u(t,x)\,\mathrm{d}x = \begin{cases} 0 & n \neq \beta m \\ (-1)^{\beta m}(\lambda t^{\alpha})^m \dfrac{\Gamma(\beta m+1)}{\Gamma(\alpha m+1)} & n = \beta m \end{cases} \tag{74}$$

where $r = 1, 2, 3, \ldots$ and $m = 0, 1, 2, \ldots$.

So we can conclude that there does not exist any moment of the fundamental solution $u(t,x)$ when $0 < \beta < n$. □

5. Special case and applications

We consider the special case $\alpha = 1/2$ of the initial value problem (1), (9) with $g(x) = \delta(x)$

$$\left({}^{C}D_t^{1/2}u\right)(t,x) = \lambda \frac{\partial u(t,x)}{\partial x} \qquad (t > 0, x \in \mathbb{R}) \tag{75}$$

$$\lim_{|x|\to\infty} u(t,x) = 0 \qquad u(t,x) = \delta(x). \tag{76}$$

According to (44) and (45), its fundamental solution is given by

$$u(t,x) = \begin{cases} 0 & x > 0 \\ \dfrac{1}{\lambda\sqrt{t}}\varphi\left(-\dfrac{1}{2}, \dfrac{1}{2}; \dfrac{x}{\lambda\sqrt{t}}\right) & x < 0 \end{cases} \tag{77}$$

for $\lambda > 0$, and by

$$u(t,x) = \begin{cases} -\dfrac{1}{\lambda\sqrt{t}}\varphi\left(-\dfrac{1}{2}, \dfrac{1}{2}; \dfrac{x}{\lambda\sqrt{t}}\right) & x > 0 \\ 0 & x < 0 \end{cases} \tag{78}$$

for $\lambda < 0$. It is directly verified that

$$\varphi\left(-\frac{1}{2}, \frac{1}{2}; z\right) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right) \tag{79}$$

and hence the fundamental solution in (77) and (78) takes the form

$$u(t,x) = \begin{cases} 0 & x > 0 \\ \dfrac{1}{\lambda\sqrt{t\pi}} \exp\left(-\dfrac{x^2}{4\lambda^2 t}\right) & x < 0 \end{cases} \tag{80}$$

and

$$u(t,x) = \begin{cases} -\dfrac{1}{\lambda\sqrt{t\pi}} \exp\left(-\dfrac{x^2}{4\lambda^2 t}\right) & x > 0 \\ 0 & x < 0 \end{cases} \tag{81}$$

for $\lambda > 0$ and $\lambda < 0$, respectively.

In accordance with (60), the moments of this fundamental solution are given by

$$\int_{-\infty}^{+\infty} x^n u(t,x)\,\mathrm{d}x = (-\lambda\sqrt{t})^n \frac{\Gamma(n+1)}{\Gamma[(n+2)/2]} \qquad (n = 0, 1, 2, \ldots). \tag{82}$$

In particular, the fundamental solution of the problem (75), (76) with $\lambda = 1$ is given by

$$u(t,x) = \begin{cases} 0 & x > 0 \\ \dfrac{1}{\sqrt{t\pi}} \exp\left(-\dfrac{x^2}{4t}\right) & x < 0 \end{cases} \tag{83}$$

while for $\lambda = -1$ by

$$u(t,x) = \begin{cases} \dfrac{1}{\sqrt{t\pi}} \exp\left(-\dfrac{x^2}{4t}\right) & x > 0 \\ 0 & x < 0. \end{cases} \tag{84}$$

By (82), the moments of the fundamental solutions of the problem (75), (76) with $\lambda = 1$ and $\lambda = -1$ are represented by

$$\int_{-\infty}^{+\infty} x^n u(t,x)\,\mathrm{d}x = (-\sqrt{t})^n \frac{\Gamma(n+1)}{\Gamma([n+2]/2)} \qquad (n = 0, 1, 2, \ldots) \tag{85}$$

and

$$\int_{-\infty}^{+\infty} x^n u(t,x)\,\mathrm{d}x = t^{n/2} \frac{\Gamma(n+1)}{\Gamma([n+2]/2)} \qquad (n = 0, 1, 2, \ldots) \tag{86}$$

respectively.

Finally, we give the application of the above results to the equation of the form (75) arising in the analysis of diffusion mechanisms with internal degrees of freedom while studying the root square of the one-dimensional diffusion equation $u_t - u_{xx} = 0$; see [18, 19].

If we use the property $\partial_t^{1/2}\partial_t^{1/2}u = \partial_t u$, being held for 'sufficiently good' functions $u(t,x)$, for instance when $u(t,x)$ is a continuous function in t, then $u_t - u_{xx} = 0$ can be rewritten in the following form [18]:

$$\left(A\partial_t^{1/2} + B\frac{\partial}{\partial x}\right)\psi(t,x) = 0 \qquad \psi(t,x) = \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} \tag{87}$$

where A and B are 2×2 matrices satisfying the conditions:

$$A^2 = I \qquad B^2 = -I \qquad AB + BA = 0 \tag{88}$$

and I is the identity operator. One of the possible choices, according to *Pauli's algebra*, is $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and, therefore, the system (87) is reduced to the Dirac-type equations

$$\partial_t^{1/2} v(t,x) + \frac{\partial}{\partial x} v(t,x) = 0 \tag{89}$$

$$\partial_t^{1/2} u(t,x) - \frac{\partial}{\partial x} u(t,x) = 0. \tag{90}$$

Now, if we put $\partial_t^{1/2} = {}^C D_t^{1/2}$, then equations (89) and (90) are special cases of equation (75) with $\lambda = -1$ and $\lambda = 1$, respectively.

Remark 4. Formulae (64) and (65) can be used to obtain the explicit solutions of more general Dirac-type decompositions of the diffusion equations.

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